Nalanda Open University

M.SC Part-1

Course : Physics

Paper : 1

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Topic- Mathematical Physics (Normal frequency of Double Pendulum)

Double Pendulum

A double pendulum is undoubtedly an actual miracle of nature. The jump in complexity, which is observed at the transition from a simple pendulum to a double pendulum is amazing. The oscillations of a simple pendulum are regular. For small deviations from equilibrium, these oscillations are harmonic and can be described by sine or cosine function. In the case of nonlinear oscillations, the period depends on the amplitude, but the regularity of the motion holds. In other words, in the case of a simple pendulum, the approximation of small oscillations fully reflects the essential properties of the system.

Double pendulum “behaves” quite differently. In the regime of small oscillations, the double pendulum demonstrates the phenomenon of beats. The character of oscillations of the pendulums changes radically with increasing energy − the oscillations become chaotic. Despite the fact that the double pendulum can be described by a system of several ordinary differential equations, that is by a completely deterministic model, the appearance of chaos looks very unusual. This situation is reminiscent of the Lorenz system where a deterministic model of three equations also shows chaotic behavior.

Normal frequency of Double Pendulum

We will build a mathematical model of the double pendulum in the form of a system of nonlinear differential equations.

In Lagrangian mechanics, evolution of a system is described in terms of the generalized coordinates and generalized velocities.

In our case, the deflection angles of the pendulums α_1 , α_2 and the angular velocities $\dot{\alpha}_1$, $\dot{\alpha}_2$ can be taken as the generalized variables.

Using these variables, we construct the Lagrangian for the double pendulum and write the Lagrange differential equations.

A simplified model of the double pendulum is shown in Figure 1.

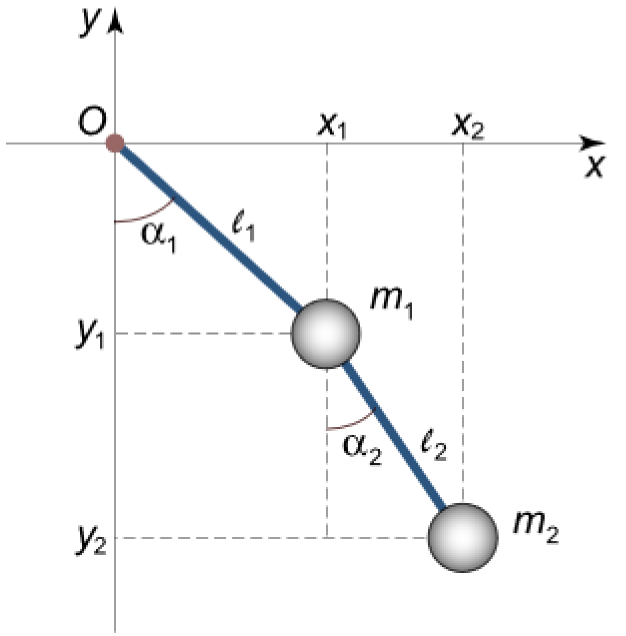

Figure 1.

We assume that the rods are massless. Their lengths are l_1 and l_2. The point masses (they are represented by the balls of finite radius) are m_1 and m_2. All pivots are assumed to be frictionless.

We introduce the xy-coordinate system, the origin O of which coincides with suspension point of the upper pendulum. The coordinates of the pendulums are defined by the following relationships:

$$x_1 = l_1 \sin\alpha_1, \;\; x_2 = l_1 \sin\alpha_1 + l_2 \sin\alpha_2, \;\; y_1 = -l_1 \cos\alpha_1, \;\; y_2 = -l_1 \cos\alpha_1 - l_2 \cos\alpha_2.$$

The kinetic and potential energy of the pendulums (respectively, T and V) are expressed by the formulas

$$T = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{m_1\left(\dot{x}_1^2 + \dot{y}_1^2\right)}{2} + \frac{m_2\left(\dot{x}_2^2 + \dot{y}_2^2\right)}{2}, \;\; V = m_1 g y_1 + m_2 g y_2.$$

Then the Lagrangian can be written as

$$L = T - V = T_1 + T_2 - (V_1 + V_2) = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) - m_1 g y_1 - m_2 g y_2.$$

Take into account that

$$\dot{x}_1 = l_1 \cos\alpha_1 \cdot \dot{\alpha}_1, \;\; \dot{x}_2 = l_1 \cos\alpha_1 \cdot \dot{\alpha}_1 + l_2 \cos\alpha_2 \cdot \dot{\alpha}_2,$$

$$\dot{y}_1 = l_1 \sin\alpha_1 \cdot \dot{\alpha}_1, \;\; \dot{y}_2 = l_1 \sin\alpha_1 \cdot \dot{\alpha}_1 + l_2 \sin\alpha_2 \cdot \dot{\alpha}_2.$$

Consequently,

$$T_1 = \frac{m_1}{2}\left(\dot{x}_1^2 + \dot{y}_1^2\right) = \frac{m_1}{2}\left(l_1^2\dot{\alpha}_1^2\cos^2\alpha_1 + l_1^2\dot{\alpha}_1^2\sin^2\alpha_1\right) = \frac{m_1}{2} l_1^2\dot{\alpha}_1^2,$$

$$T_2 = \frac{m_2}{2}\left(\dot{x}_2^2 + \dot{y}_2^2\right) = \frac{m_2}{2}\Big[\left(l_1\dot{\alpha}_1\cos\alpha_1 + l_2\dot{\alpha}_2\cos\alpha_2\right)^2$$
$$+ \left(l_1\dot{\alpha}_1\sin\alpha_1 + l_2\dot{\alpha}_2\sin\alpha_2\right)^2\Big] = \frac{m_2}{2}\Big[l_1^2\dot{\alpha}_1^2\cos^2\alpha_1 + l_2^2\dot{\alpha}_2^2\cos^2\alpha_2$$
$$+ 2l_1l_2\dot{\alpha}_1\dot{\alpha}_2\cos\alpha_1\cos\alpha_2 + l_1^2\dot{\alpha}_1^2\sin^2\alpha_1 + l_2^2\dot{\alpha}_2^2\sin^2\alpha_2 + 2l_1l_2\dot{\alpha}_1\dot{\alpha}_2\sin\alpha_1\sin\alpha_2\Big]$$
$$= \frac{m_2}{2}\left[l_1^2\dot{\alpha}_1^2 + l_2^2\dot{\alpha}_2^2 + 2l_1l_2\dot{\alpha}_1\dot{\alpha}_2\cos(\alpha_1 - \alpha_2)\right],$$

$$V_1 = m_1 g y_1 = -m_1 g l_1 \cos\alpha_1,$$

$$V_2 = m_2 g y_2 = -m_2 g\left(l_1\cos\alpha_1 + l_2\cos\alpha_2\right).$$

As a result, the Lagrangian of the system takes the following form:

$$L = T - V = T_1 + T_2 - (V_1 + V_2) = \left(\frac{m_1}{2} + \frac{m_2}{2}\right) l_1^2 \dot{\alpha}_1^2 + \frac{m_2}{2} l_2^2 \dot{\alpha}_2^2$$
$$+ m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + (m_1 + m_2)\, g l_1 \cos\alpha_1 + m_2 g l_2 \cos\alpha_2.$$

Now we can write the Lagrange equations (sometimes they are called as the Euler-Lagrange equations):

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_i} - \frac{\partial L}{\partial \alpha_i} = 0,\;\; i = 1, 2.$$

The partial derivatives in these equations are expressed by the following formulas:

$$\frac{\partial L}{\partial \dot{\alpha}_1} = (m_1 + m_2)\, l_1^2 \dot{\alpha}_1 + m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2),$$

$$\frac{\partial L}{\partial \alpha_1} = -m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - (m_1 + m_2)\, g l_1 \sin\alpha_1,$$

$$\frac{\partial L}{\partial \dot{\alpha}_2} = m_2 l_2^2 \dot{\alpha}_2 + m_2 l_1 l_2 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2),$$

$$\frac{\partial L}{\partial \alpha_2} = m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \sin(\alpha_1 - \alpha_2) - m_2 g l_2 \sin\alpha_2$$

Hence, the first Lagrange equation can be written as

$$\frac{d}{dt}\left[(m_1+m_2)\,l_1^2\dot{\alpha}_1+m_2l_1l_2\dot{\alpha}_2\cos(\alpha_1-\alpha_2)\right]+m_2l_1l_2\dot{\alpha}_1\dot{\alpha}_2\sin(\alpha_1-\alpha_2)$$
$$+\,(m_1+m_2)\,gl_1\sin\alpha_1=0,$$

$$\Rightarrow (m_1+m_2)\,l_1^2\ddot{\alpha}_1+m_2l_1l_2\ddot{\alpha}_2\cos(\alpha_1-\alpha_2)+m_2l_1l_2\dot{\alpha}_2^2\sin(\alpha_1-\alpha_2)$$
$$+\,(m_1+m_2)\,gl_1\sin\alpha_1=0.$$

Cancelling $l_1\neq 0$, we obtain:

$$(m_1+m_2)\,l_1\ddot{\alpha}_1+m_2l_2\ddot{\alpha}_2\cos(\alpha_1-\alpha_2)+m_2l_2\dot{\alpha}_2^2\sin(\alpha_1-\alpha_2)+(m_1+m_2)\,g\sin\alpha_1=0.$$

Similarly, we derive the second differential equation:

$$\frac{d}{dt}\left[m_2l_2^2\dot{\alpha}_2+m_2l_1l_2\dot{\alpha}_1\cos(\alpha_1-\alpha_2)\right]-m_2l_1l_2\dot{\alpha}_1\dot{\alpha}_2\sin(\alpha_1-\alpha_2)+m_2gl_2\sin\alpha_2=0,$$

$$\Rightarrow m_2l_2^2\ddot{\alpha}_2+m_2l_1l_2\ddot{\alpha}_1\cos(\alpha_1-\alpha_2)-m_2l_1l_2\dot{\alpha}_1^2\sin(\alpha_1-\alpha_2)+m_2gl_2\sin\alpha_2=0.$$

After canceling $m_2l_2\neq 0$ the equation takes the form

$$l_2\ddot{\alpha}_2+l_1\ddot{\alpha}_1\cos(\alpha_1-\alpha_2)-l_1\dot{\alpha}_1^2\sin(\alpha_1-\alpha_2)+g\sin\alpha_2=0.$$

Thus, the nonlinear system of two Lagrange differential equations can be written as

$$(m_1 + m_2)\, l_1 \ddot{\alpha}_1 + m_2 l_2 \ddot{\alpha}_2 \cos(\alpha_1 - \alpha_2) + m_2 l_2 \dot{\alpha}_2^2 \sin(\alpha_1 - \alpha_2) + (m_1 + m_2)\, g \sin \alpha_1 = 0.$$

$$l_2 \ddot{\alpha}_2 + l_1 \ddot{\alpha}_1 \cos(\alpha_1 - \alpha_2) - l_1 \dot{\alpha}_1^2 \sin(\alpha_1 - \alpha_2) + g \sin \alpha_2 = 0.$$

Small Oscillations of the Double Pendulum

Assuming that the angles $\alpha_1(t), \alpha_2(t)$ are small, the oscillations of the pendulums near the zero equilibrium point can be described by a linear system of equations. To get such a system, let's get back to the original Lagrangian of the system:

$$L = T - V = \left(\frac{m_1}{2} + \frac{m_2}{2}\right) l_1^2 \dot{\alpha}_1^2 + \frac{m_2}{2} l_2^2 \dot{\alpha}_2^2 + m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 \cos(\alpha_1 - \alpha_2)$$
$$+ (m_1 + m_2) g l_1 \cos\alpha_1 + m_2 g l_2 \cos\alpha_2.$$

We write this Lagrangian in a simpler form, expanding it in a Maclaurin series and retaining the linear and quadratic terms. The trigonometric functions can be replaced by the following approximate expressions:

$$\cos\alpha_1 \approx 1 - \frac{\alpha_1^2}{2}, \;\; \cos\alpha_2 \approx 1 - \frac{\alpha_2^2}{2}, \;\; \cos(\alpha_1 - \alpha_2) \approx 1 - \frac{(\alpha_1 - \alpha_2)^2}{2} \approx 1.$$

Here we have taken into account that the term with $\cos(\alpha_1 - \alpha_2)$ contains the product of small quantities $\dot{\alpha}_1 \dot{\alpha}_2$ and has the second order of smallness. Therefore, we can leave only the linear term in the cosine expansion.

Substituting this in the original Lagrangian and considering that the potential energy is defined up to a constant, we obtain the quadratic Lagrangian for the double pendulum in the form:

$$L = T - V = \left(\frac{m_1}{2} + \frac{m_2}{2}\right) l_1^2 \dot{\alpha}_1^2 + \frac{m_2}{2} l_2^2 \dot{\alpha}_2^2 + m_2 l_1 l_2 \dot{\alpha}_1 \dot{\alpha}_2 - \left(\frac{m_1}{2} + \frac{m_2}{2}\right) g l_1 \alpha_1^2$$
$$+ \frac{m_2}{2} g l_2 \alpha_2^2.$$

We derive the Lagrange differential equations for the given Lagrangian. They are written as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}_1} - \frac{\partial L}{\partial \alpha_1} = 0, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{\alpha}_2} - \frac{\partial L}{\partial \alpha_2} = 0.$$

Find the partial derivatives:

$$\frac{\partial L}{\partial \dot{\alpha}_1} = (m_1 + m_2) \, l_1^2 \dot{\alpha}_1 + m_2 l_1 l_2 \dot{\alpha}_2,$$

$$\frac{\partial L}{\partial \alpha_1} = -\left(m_1 + m_2\right) g l_1 \alpha_1,$$

$$\frac{\partial L}{\partial \dot{\alpha}_2} = m_2 l_2^2 \dot{\alpha}_2 + m_2 l_1 l_2 \dot{\alpha}_1,$$

$$\frac{\partial L}{\partial \alpha_2} = -m_2 g l_2 \alpha_2.$$

We get the system of two differential equations

$$\frac{d}{dt}\left[(m_1+m_2)\,l_1^2\dot{\alpha}_1+m_2l_1l_2\dot{\alpha}_2\right]+(m_1+m_2)\,gl_1\alpha_1=0,$$

$$\frac{d}{dt}\left[m_2l_2^2\dot{\alpha}_2+m_2l_1l_2\dot{\alpha}_1\right]+m_2gl_2\alpha_2=0.$$

or

$$(m_1+m_2)\,l_1^2\ddot{\alpha}_1+m_2l_1l_2\ddot{\alpha}_2+(m_1+m_2)\,gl_1\alpha_1=0,$$

$$m_2l_1l_2\ddot{\alpha}_1+m_2l_2^2\ddot{\alpha}_2+m_2gl_2\alpha_2=0.$$

This system of equations can be written in a compact matrix form. We introduce the matrices

$$\boldsymbol{\alpha}\,(t)=\begin{bmatrix}\alpha_1\,(t)\\ \alpha_2\,(t)\end{bmatrix},\;M=\begin{bmatrix}(m_1+m_2)\,l_1^2 & m_2l_1l_2\\ m_2l_1l_2 & m_2l_2^2\end{bmatrix},$$
$$K=\begin{bmatrix}(m_1+m_2)\,gl_1 & 0\\ 0 & m_2gl_2\end{bmatrix},\;\mathbf{0}=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

Then the system of differential equations can be represented as

$$M\ddot{\boldsymbol{\alpha}}+K\boldsymbol{\alpha}=\mathbf{0}.$$

In the case of one body, this equation describes the free undamped oscillations with a certain frequency. In the case of the double pendulum, the solution (as you will see below) will contain oscillations with two characteristic frequencies, which are called normal modes. The normal modes are the real part of the complex-valued vector function

$$\boldsymbol{\alpha}(t) = \begin{bmatrix} \alpha_1(t) \\ \alpha_2(t) \end{bmatrix} = \text{Re}\left(\begin{bmatrix} \mathbf{H}_1 \\ \mathbf{H}_2 \end{bmatrix} e^{i\omega t} \right),$$

where $\mathbf{H}_1$, $\mathbf{H}_2$ are the eigenvectors, ω is the real frequency. The values of the normal frequencies $\omega_{1,2}$ are determined by solving the auxiliary equation

$$\det\left(K - \omega^2 M\right) = 0.$$

In the case of arbitrary masses m_1, m_2 and lengths l_1, l_2, the auxiliary equation takes the form

$$(m_1 + m_2)\, g^2 - \omega^2 (m_1 + m_2)(l_1 + l_2)\, g + \omega^4 m_1 l_1 l_2 = 0.$$

Thus, we have a biquadratic equation for the frequencies ω. The general solution for this equation is somewhat cumbersome. Therefore we consider the case when the lengths of the rods of both pendulums are equal: $l_1 = l_2 = l$. Then the normal frequencies will be determined by a compact formula

$$\omega_{1,2}^2 = \, = \frac{g}{l}\left[1 + \mu \pm \sqrt{(1+\mu)\,\mu}\right], \;\; \text{where} \;\; \mu = \frac{m_2}{m_1}.$$

As it can be seen, the eigenfrequencies $\omega_{1,2}$ depend only on the mass ratio $\mu = \frac{m_2}{m_1}$. The dependencies of the frequencies ω_1, ω_2 on the parameter μ (provided $\frac{g}{l} = 1$) are shown in Figure 2.

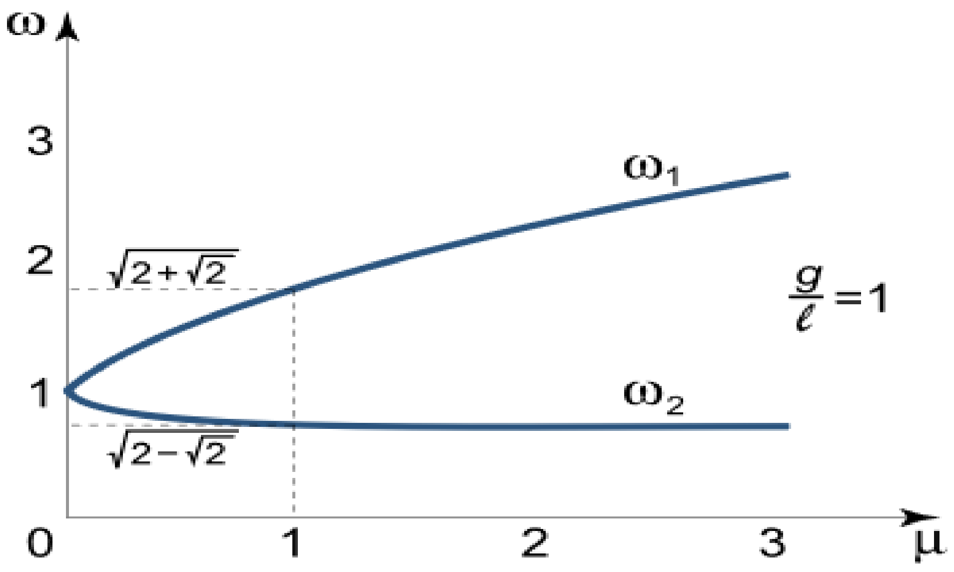

Figure 2.

For equal masses $m_1 = m_2 = m$, that is when $\mu = 1$, the frequencies are given by

$$\boxed{\omega_{1,2} = \sqrt{\frac{g}{l}}\sqrt{2 \pm \sqrt{2}}.}$$