ROOTS OF NON-LINEAR EQUATIONS

Lecture 2: Regula Falsi method.

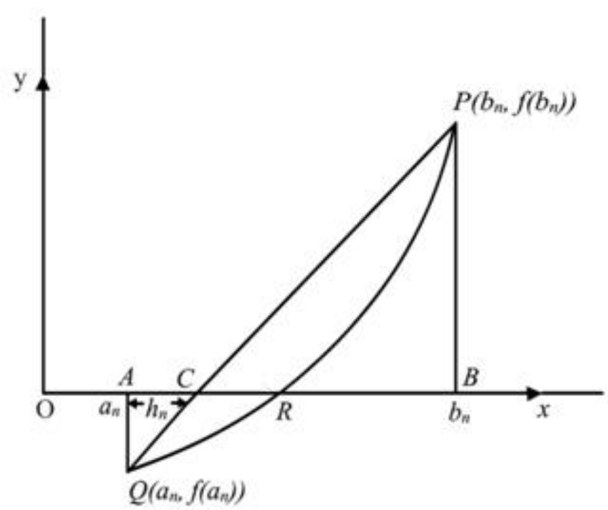
4. **Regula-Falsi Method or Method of False Position**

In this method, we first find a sufficiently small interval $[a_o, b_o]$, such that $f(a_0)f(b_o) < 0$, by tabulation or graphical method, and which contains only one root α (say) of $f(x) = 0, i.e. f'(x)$ maintains same sign in $[a_o, b_o]$.

This method is based on the assumption that the graph of $y = f(x)$ in the small interval $[a_o, b_o]$ can be represented by the chord joining $(a_{0,}\ f(a_0))$ and $\left(b_{o,}\ f\left(b_{o,}\right)\right)$. Therefore, at the point $x = x_1 = a_0 + h_0$, at which the chord meets the x-axis, we obtain two intervals $[a_o, x_1]$ and $[x_1, b_o]$, one of which must contain the root α, depending upon the condition $f(a_o)f(x_1) < 0$ or $f\left(x_{1,}\right)f(b_0) < 0$.

$Let\ f\left(x_{1,}\right)f(b_0) < 0$, then α lies in the interval $[x_{1,} b_o]$ which we rename as $[a_{1,} b_1]$. Again, we consider that the graph of $y = f(x)$ in $[a_{1,} b_1]$ as the chord joining $(a_{1,} f(a_1))$ and $(b_{1,} f(b_1))$. Thus, the point of intersection of the chord with the x-axis (say) $x_2 = a_1 + h_1$ gives us an approximate value of the root α of the equation $f(x) = 0$.

Now we are going to establish an iteration formula which may generate a sequence of successive approximations of an exact root α of the equation $f(x) = 0$, Geometrically, we interpret it as follows:



In the above figure, we assume that one root α of $f\ (x) = 0$ lies in the small interval $[a_n\ , b_n]$ and $f(a_n) < 0$ and $f(b_n) > 0$. Let *PRQ* be the graph of $y = f(x)$ in $[a_{n,} b_n]$ intersecting the x-axis at R.

Thus, $x = OR\ (= \alpha\ )$ gives the exact value of the root α. If we consider the curve *PRQ* as the chord *PQ*, in the small interval $\left[a_{n,} b_n\right]$, which intersects the x-axis at C, then OC = $x_{n+1}\ = a_n\ + h_n$ approximates the root α of the equation $f(x)$=0.

Now from similar triangles *AQC* and *CBP*, we get,

$$\frac{AC}{AQ} = \frac{CB}{BP} \text{ or } AC = \frac{AQ}{BP} CB = \frac{|f(a_n)|}{|f(b_n)|} (AB - AC)$$

$$\text{or, } AC\left[1 + \frac{|f(a_n)|}{|f(b_n)|}\right] = \frac{|f(a_n)|}{|f(b_n)|} \cdot AB = \frac{|f(a_n)|}{|f(b_n)|} (b_n - a_n)$$

$$\therefore \quad AC = h_n = \frac{|f(a_n)|}{|f(a_n)| + |f(b_n)|} (b_n - a_n).$$

$$\text{Thus,} \quad x_{n+1} = a_n + h_n = a_n + \frac{|f(a_n)|}{|f(a_n)| + |f(b_n)|} \cdot (b_n - a_n).$$

The above formula is known as the iteration formula for Regula-Falsi method.

Example 3. Find a root of the equation $3x - \cos x - 1 = 0$ by Regula-Falsi method, correct to four significant figures.

Solution: Let $f(x) = 3x - \cos x - 1$. As f(0) = –1 and f(1) = 1.46, f(x) = 0 has a root between 0 and 1. Now we compute the root as follows:

n	$a_n(-)$	$b_n(+)$	$f(a_n)$	$f(b_n)$	h_n	x_{n+1}	$f(x_{n+1})$
0	0.0	1.0	-1	1.46	0.41	0.41	-0.67 <0
1	0.41	1.0	-0.67	1.46	0.18	0.59	-0.061 < 0
2	0.59	1.0	-0.061	1.46	0.0164	0.6064	-0.0025 < 0
3	0.6064	1.0	-0.0025	1.46	0.00067	0.60707	-0.000113 < 0
4	0.60707	1.0	-0.000113	1.46	0.0000304	0.6071004	-0.0000045 <0
5	0.6071004	1.0	-0.0000045	1.46	0.0000012	0.6071016	-0.00000017 < 0

$$\text{Here, } h_n = \frac{|f(a_n)|(b_n - a_n)}{|f(a_n)| + |f(b_n)|}; \; x_{n+1} = a_n + h_n.$$

Thus, 0.6071 is a root of $f(x) = 0$, is correct up to four significant figures.

Example 4: Compute a root of x ln(x) = 1 by Regula-Falsi Method, correct to three decimal places.

Solution: Let f(x) = x ln(x) – 1. Here $f(1) = -1$ and $f(2) = 0.39$, therefore, f(x) = 0 has a root between 1 and 2. Now we compute the successive approximations of the root as follows:

n	$a_n(-)$	$b_n(+)$	$f(a_n)$	$f(b_n)$	h_n	x_{n+1}	$f(x_{n+1})$
0	1.0	2.0	-1.0	0.39	0.72	1.72	-0.067 < 0
1	1.72	2.0	-0.067	0.39	107611	1.7611	-0.00333 < 0
2	1.7611	2.0	-0.00333	0.39	0.002022	1.763122	-0.000158 < 0
3	1.763122	2.0	-0.000158	0.39	0.000096	1.763218	-0.0000075 < 0

$$\text{Here, } h_n = \frac{|f(a_n)|(b_n - a_n)}{|f(a_n)| + |f(b_n)|};\ x_{n+1} = a_n + h_n.$$

Thus, **1.763** is root of f(x) = 0, correct up to three decimal places.

Example 5. Compute the root of the equation $2x - log_{10}x - 7 = 0$, by Regula-Falsi method, which lies between 3 and 4, correct to three decimal places.

Solution. Let $f\ (x) = 2x - log_{10}x - 7$.
Here $f(3) = -1.48, f(4) = 0.40$. Therefore, one root of $f\ (x) = 0$ lies between 3 and 4. Now we compute the successive approximations of the root as follows:

n	a_n(-)	b_n(+)	$f(a_n)$	$f(b_n)$	h_n	x_{n+1}	$f(x_{n+1})$
0	3.0	4.0	-1.48	0.40	0.79	3.79	0.0014>0
1	3.0	3.79	-1.48	0.0014	0.789	3.789	-0.00052<0
2	3.789	3.79	-0.00052	0.0014	0.000271	3.789271	-0.0000014<0
3	3.789271	3.79	-0.0000014	0.0014	0.000007	3.7892717	-0.0000012<0

$$\text{Here, } h_n = \frac{|f(a_n)|(b_n - a_n)}{|f(a_n)| + |f(b_n)|};\ x_{n+1} = a_n + h_n.$$

Thus, **3.789** is a root of $f(x) = 0$,correct to three decimal places. In iteration number 2, we can conclude that **3.79** in a root, correct up to two decimal places.

Exercise 3. Find a real root of $x^x + 2x - 6 = 0$ by Regula Falsi method, correct to three decimal places. (Ans. 1.723)

Exercise 4. Find a real root of x = sin(x) + 0.25 by Regula Falsi method, correct to three decimal places. (Ans. 1.172)