SOLUTION OF SYSTEM OF LINEAR EQUATIONS

Lecture 4: (a) Jacobi's method.
method (general).
(b) Gauss Seidel method.

Jacobi's Method:

Carl Gustav Jacob Jacobi (1804-1851) gave an indirect method for finding the solution of a system of linear equations, which is based on the successive better approximations of the values of the unknowns, using an iterative procedure. The sufficient condition for the convergence of Gauss Jacobi method to solve $A\tilde{x} = \underset{\sim}{b}$ is that the coefficient matrix A is strictly diagonally row dominant, that is, if

$$A = \begin{pmatrix} a_{11} & a_{12} & -- & a_{1n} \\ a_{21} & a_{22} & -- & a_{2n} \\ - & - & - & - \\ a_{n_1} & a_{n_2} & -- & a_{n_n} \end{pmatrix}, \text{ then}$$

$$|a_{ii}| > \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

It should be noted that this method makes two assumptions. First, the system of linear equations to be solved, must have a unique solution and second, there should not be any zeros on the main diagonal of the coefficient matrix A. In case, there exist zeros on its main diagonal, then rows must be interchanged to obtain a coefficient matrix that does not have zero entries on the main diagonal.

Consider a system of n linear equations in n unknowns, which are strictly diagonally row dominant, as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1,n-1}x_{n-1} + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2,n-1}x_{n-1} + a_{2n}x_n &= b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots\dots\dots + a_{n,n-1}x_{n-1} + a_{nn}x_n &= b_n \end{aligned}$$

Since the system is strictly diagonally row dominant, $a_{ii} \neq 0$.

Therefore, the system of equations is rewritten as

$$x_1 = \frac{b_1}{a_{11}} - 0.x_1 - \frac{a_{12}}{a_{11}} x_2 - .... - \frac{a_{1,n-1}}{a_{11}} x_{n-1} - \frac{a_{1n}}{a_{11}} x_n$$

$$x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1 - 0.x_2 - .... - \frac{a_{2,n-1}}{a_{22}} x_{n-1} - \frac{a_{2n}}{a_{22}} x_n$$

$$...$$

$$x_n = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1 - \frac{a_{n2}}{a_{nn}} x_2 - .... - \frac{a_{n,n-1}}{a_{nn}} x_{n-1} - 0.x_n$$

We then consider an arbitrary initial guess of the solution as

$$\left(x_1^{(0)}, x_2^{(0)}, ..., x_n^{(0)}\right) ,$$

which are row substituted to the right hand side of the rewritten equations to obtain the first approximation as

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - .... - a_{1n-1}x_{n-1}^{(0)} - a_{1n}x_n^{(0)}\right]$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)} - .... - a_{2,n-1}x_{n-1}^{(0)} - a_{2n}x_n^{(0)}\right]$$

$$-----------------------------------$$

$$x_n^{(1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(0)} - a_{n2}x_2^{(0)} - .... - a_{n,n-1}x_{n-1}^{(0)}\right]$$

This process is repeated by substituting the first approximate solution $\left(x_1^{(1)}, x_2^{(1)}, —, x_n^{(1)}\right)$ to the r.h.s of the rewritten equations. By repeated iteration, we get the required solution up to the desired level of the accuracy.

Example 6. Solve the system of linear equations by Jacobi's method.

$$\begin{aligned} x_1 + x_2 + 4x_3 &= 9 \\ 8x_1 - 3x_2 + 2x_3 &= 20 \\ 4x_1 + 11x_2 - x_3 &= 33 \end{aligned}$$

<u>Solution</u>: The given system of equations is not diagonally row dominant as $|a_{11}| < |a_{12}| + |a_{13}|$. Therefore, we re-arrange the system as

$$\begin{aligned} 8x_1 - 3x_2 + 2x_3 &= 20 \\ 4x_1 + 11x_2 - x_3 &= 33 \\ x_1 + x_2 + 4x_3 &= 9 \end{aligned}$$

Here, $|8| > |-3| + |2|, |11| > |4| + |-1|$ and $|4| > |1| + |1|$. Thus, the system is diagonally row dominant. We now re-write the system as

$$\begin{aligned} x_1 &= \frac{1}{8}(20 + 3x_2 - 2x_3) \\ x_2 &= \frac{1}{11}(33 - 4x_1 + x_3) \\ x_3 &= \frac{1}{4}(9 - x_1 - x_2) \end{aligned}$$

Let the initial guess be $x_1^{(0)} = 1, x_2^{(0)} = 1, x_3^{(0)} = 0$ .Then, the first approximation to the solution is given by

$$\begin{cases} x_1^{(1)} = \frac{1}{8}(20 + 3\times1 - 2\times0) = 2.875 \\ x_2^{(1)} = \frac{1}{11}(33 - 4\times1 + 0) = 2.636 \\ x_3^{(1)} = \frac{1}{4}(9 - 1 - 1) = 1.75 \end{cases}$$

Second approximation

$$\begin{cases} x_1^{(2)} &= \frac{1}{8}\left(20+3\times 2.636-2\times 1.75\right)=3.051 \\ x_2^{(2)} &= \frac{1}{11}\left(33-4\times 2.875+1.75\right)=2.114 \\ x_3^{(3)} &= \frac{1}{4}\left(9-2.875-2.636\right)=0.872 \end{cases}$$

Third approximation

$$\begin{aligned} x_1^{(3)} &= 3.075 \\ x_2^{(3)} &= 1.969 \\ x_3^{(3)} &= 0.959 \end{aligned}$$

Fourth approximation

$$\begin{cases} x_1^{(4)} &= 2.999 \\ x_2^{(4)} &= 1.969 \\ x_3^{(4)} &= 0.989 \end{cases}$$

Fifth approximation

$$\begin{cases} x_1^{(5)} &= 2.991 \\ x_2^{(5)} &= 1.999 \\ x_3^{(5)} &= 1.008 \end{cases}$$

Sixth approximation

$$\begin{cases} x_1^{(6)} &= 2.997 \\ x_2^{(6)} &= 2.004 \\ x_3^{(6)} &= 1.002 \end{cases}$$

Therefore, $x_1 = 3.0, x_2 = 2.0$ and $x_3 = 1.0,$ correct to two significant figures.

Gauss Seidel Method

Gauss Seidel iteration method for solving a system of n-linear equations in n-unknowns is a modified Jacobi's method. Therefore, all the conditions that is true for Jacobi's method, also holds for Gauss Seidel method. As before, the system of linear equations are rewritten as

$$x_1 = \frac{b_1}{a_{11}} - 0.x_1 - \frac{a_{12}}{a_{11}} x_2 - \ldots\ldots - \frac{a_{1,n-1}}{a_{11}} x_{n-1} - \frac{a_{1n}}{a_{11}} x_n$$

$$x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1 - 0.x_2 - \ldots\ldots - \frac{a_{2,n-1}}{a_{22}} x_{n-1} - \frac{a_{2n}}{a_{22}} x_n$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_n = \frac{b_n}{a_{nn}} - \frac{a_{n1}}{a_{nn}} x_1 - \frac{a_{n2}}{a_{nn}} x_2 - \ldots\ldots - \frac{a_{n,n-1}}{a_{nn}} x_{n-1} - 0.x_n$$

If $\left(x_1^{(0)}, x_2^{(0)}, \ldots\ldots, x_n^{(0)}\right)$ be the initial guess of the solution, which is arbitrary, then the first approximation to the solution is obtained as

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)} - \ldots\ldots - a_{1,n-1}x_{n-1}^{(0)} - a_{1n}x_n^{(0)}\right]$$

$$x_2^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)} - \ldots\ldots - a_{2,n-1}x_{n-1}^{(0)} - a_{2n}x_n^{(0)}\right]$$

$$x_3^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)} - a_{34}x_4^{(0)} - \ldots\ldots - a_{3,n-1}x_{n-1}^{(0)} - a_{3n}x_n^{(0)}\right]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$x_n^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{n1}x_1^{(1)} - a_{n2}x_2^{(1)} - \ldots\ldots\ldots - a_{n,n-1}x_{n-1}^{(1)}\right]$$

Please note, while calculating $x_2^{(1)}$, the value of x_1 is replaced by $x_1^{(1)}$, not by $x_1^{(0)}$. . This is the basic difference of Gauss Seidel with Jocobi's method.

The successive iteration's are generated by the scheme called iteration formulae of Gauss-Seidel method, which is as follows:

$$x_1^{(\kappa+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(\kappa)} - a_{13}x_3^{(\kappa)} - .... - a_{1n-1}x_{n-1}^{(\kappa)} - a_{1n}x_n^{(\kappa)}\right]$$

$$x_2^{(\kappa+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(\kappa+1)} - a_{23}x_3^{(\kappa)} - .... - a_{2n-1}x_{n-1}^{(\kappa)} - a_{2n}x_n^{(\kappa)}\right]$$

....

$$x_{n-1}^{(\kappa+1)} = \frac{1}{a_{n-1,n-1}}\left[b_{n-1} - a_{n-1,1}x_1^{(\kappa+1)} - .... - a_{n-1,n-2}x_{n-2}^{(\kappa+1)} - a_{n-1,n}x_n^{(\kappa)}\right]$$

$$x_n^{(\kappa+1)} = \frac{1}{a_{nn}}\left[b_n - a_{n1}x_1^{(\kappa+1)} - a_{n2}x_2^{(\kappa+1)} - .... - a_{n,n-1}x_{n-1}^{(\kappa+1)}\right]$$

The number of iterations (κ) required depends upon the desired degree of accuracy.

Example 6. Solve the system of linear equations by Gauss Seidel method.

$$x_1 + x_2 + 4x_3 = 9$$
$$8x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$

<u>Solution</u>: The given system of equations is not diagonally row dominant as $|a_{11}| < |a_{12}| + |a_{13}|$. Therefore, we re-arrange the system as

$$8x_1 - 3x_2 + 2x_3 = 20$$
$$4x_1 + 11x_2 - x_3 = 33$$
$$x_1 + x_2 + 4x_3 = 9$$

Here, $|8| > |-3| + |2|, |11| > |4| + |-1|$ and $|4| > |1| + |1|$. Thus, the system is diagonally row dominant. We now re-write the system as

$$x_1 = \frac{1}{8}(20 + 3x_2 - 2x_3)$$
$$x_2 = \frac{1}{11}(33 - 4x_1 + x_3)$$
$$x_3 = \frac{1}{4}(9 - x_1 - x_2)$$

Let the initial guess be $x_1^{(0)} = 1, x_2^{(0)} = 1, x_3^{(0)} = 0$. Then, the first approximation to the solution is given by

$$x_1^{(1)} = \frac{1}{8}(20 + 3\times 1 - 2\times 0) = 2.875$$
$$x_2^{(1)} = \frac{1}{11}\left(33 - 4\times x_1^{(1)} + x_3^{(0)}\right) = \frac{1}{11}(33 - 4\times 2.875 + 0) = 1.955$$
$$x_3^{(1)} = \frac{1}{4}\left(9 - x_1^{(1)} - x_2^{(1)}\right) = \frac{1}{4}(9 - 2.875 - 1.955) = 1.043$$

2nd approximation

$$\begin{cases} x_1^{(2)} &= \frac{1}{8}\left(20+3\times x_2^{(1)}-2\times x_3^{(1)}\right)=\frac{1}{8}\left(20+3\times 1.955-2\times 1.043\right)=2.972 \\ x_2^{(2)} &= \frac{1}{11}\left(33-4\times x_1^{(2)}+x_1^{(3)}\right)=\frac{1}{11}\left(33-4\times 2.972+1.043\right)=2.014 \\ x_3^{(2)} &= \frac{1}{4}\left(9-x_2^{(2)}-x_3^{(2)}\right)=\frac{1}{4}\left(9-2.972-2.014\right)=1.004 \end{cases}$$

3rd approximation

$$\begin{cases} x_1^{(3)} &= \frac{1}{8}\left(20+3\times x_2^{(2)}-2\times x_3^{(2)}\right)=\frac{1}{8}\left(20+3\times 2.014-2\times 1.004\right)=3.004 \\ x_2^{(3)} &= \frac{1}{11}\left(33-4\times x_1^{(3)}+x_3^{(2)}\right)=\frac{1}{11}\left(33-4\times 3.004+1.004\right)=1.999 \\ x_3^{(3)} &= \frac{1}{4}\left(9-x_1^{(3)}-x_2^{(3)}\right)=\frac{1}{4}\left(9-3.004-1.999\right)=0.999 \end{cases}$$

4th approximation

$$\begin{cases} x_1^{(4)} &= 3.00 \\ x_2^{(4)} &= 2.00 \\ x_3^{(4)} &= 1.00 \end{cases}$$

Therefore, $x_1 = 3.0, x_2 = 2.0$ and $x_3 = 1.0$, correct to two significant figure.

Exercises

(1) Use Jacobi's method to solve the following system of equations, with $x^{(0)} = (1,1,1)^T$ as initial approximation, correct to 2 significant figures.

$$\begin{aligned} x - 10y + 3z &= 39 \\ 10x - 2y - 5z &= 26 \\ 4x - 5y + 10z &= 47 \end{aligned}$$

What is the minimum number of iterations required to get 5 significant digit accuracy, if 5 digit arithmetic is used.

(Ans: True solution $(3,-3,2)^T$; number of iteration required=36)

(2) Do three iterations of Jacobi's method to solve

$$\begin{aligned} -2x + 3y + 10z &= 22 \\ 10x + 2y + z &= 9 \\ x + 10y - z &= -22 \end{aligned}$$

with $x^{(0)} = (1,-1,1)^T$ as starting vector. What is the minimum number of iterations required, so that the solution is correct to 4 decimal places.

(Ans: True solution $(1,-2,3)^T$; number of iteration required =17)

(3) Solve, by Gauss-Seidal iteration method, the system of linear equations

$$\begin{aligned} 3x + 9y - 2z &= 11 \\ 4x + 2y + 13z &= 24 \\ 4x - 2y + z &= -8 \end{aligned}$$

correct up to four significant figures.

(Ans: $x = -1.423, y = 2.131, z = 1.956$)

(4) Compute the solution of the system of linear equations by Gauss-Seidal iteration method

$$6.7x + 1.1y + 2.2z = 20.5$$
$$3.1x + 9.4y - 1.5z = 22.9$$
$$2.1x - 1.5y + 8.4z = 28.8$$

correct up to 3-significant figures.

(Ans: $x = 1.50, y = 2.50, z = 3.50$)

(5) Do five iterations of each Jacobi's and Gauss Seidel method to solve

$$2x + 3y + 7z = 16$$
$$3x + y + z = 6$$
$$x + 5y + 3z = 10$$

with starting initial guess as (x, y, z) = (1,1,1). What is the minimum number of iterations required, so that the solutions correct to 8 significant figures?

(Ans: True solution: $x = 1.2, y = 0.8, z = 1.6$)