Nalanda Open University

B.SC Part-1

Course : Physics(Hons)

Paper : 1

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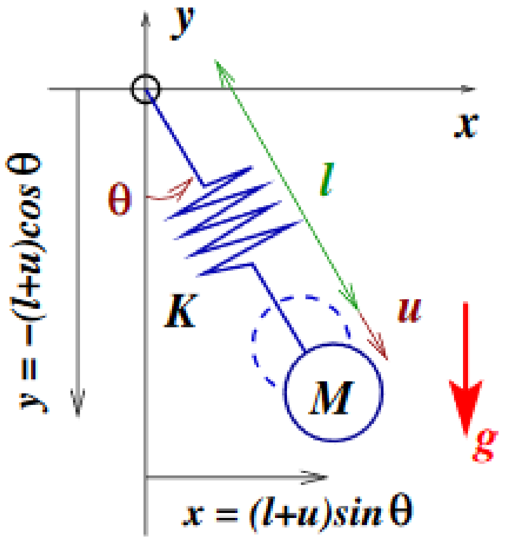
Topic- Generalized Coordinates

Cartesian Coordinates and Generalized Coordinates

The set of coordinates used to describe the motion of a dynamic system is not unique. For example, consider an elastic pendulum (a mass on the end of a spring). The position of the mass at any point in time may be expressed in Cartesian coordinates $(x(t), y(t))$ or in terms of the angle of the pendulum and the stretch of the spring $(\theta(t), u(t))$. Of course, these two coordinate systems are related. For Cartesian coordinates centered at the pivot point,



$$x(t) = (l + u(t)) \sin \theta(t) \tag{1}$$
$$y(t) = -(l + u(t)) \cos \theta(t) \tag{2}$$

where l is the un-stretched length of the spring. Let's define

$$\boldsymbol{r}(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix} = \begin{bmatrix} x(\theta(t), u(t)) \\ y(\theta(t), u(t)) \end{bmatrix} = \begin{bmatrix} (l + u(t)) \sin \theta(t) \\ -(l + u(t)) \cos \theta(t) \end{bmatrix} \tag{3}$$

and

$$\boldsymbol{q}(t) = \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ u(t) \end{bmatrix} \tag{4}$$

so that $\boldsymbol{r}(t)$ is a function of $\boldsymbol{q}(t)$.

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In Cartesian coordinates, the velocities are

$$\dot{\boldsymbol{r}}(t) = \begin{bmatrix} \dot{r}_1(t) \\ \dot{r}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{u}(t)\sin\theta(t) + (l + u(t))\cos\theta(t)\dot{\theta}(t) \\ -\dot{u}(t)\cos\theta(t) + (l + u(t))\sin\theta(t)\dot{\theta}(t) \end{bmatrix} \tag{5}$$

So, *in general*, Cartesian velocities $\dot{\boldsymbol{r}}(t)$ can be a function of both the velocity *and* position of some other coordinates ($\dot{\boldsymbol{q}}(t)$ *and* $\boldsymbol{q}(t)$). Such coordinates $\boldsymbol{q}$ are called *generalized coordinates*. The kinetic energy, T, may be expressed in terms of either $\dot{\boldsymbol{r}}$ or, *more generally*, in terms of $\dot{\boldsymbol{q}}$ and $\boldsymbol{q}$.

Small changes (or *variations*) in the rectangular coordinates, $(\delta x, \delta y)$ *consistent with all displacement constraints*, can be found from variations in the generalized coordinates $(\delta\theta, \delta u)$.

$$\delta x = \frac{\partial x}{\partial \theta}\delta\theta + \frac{\partial x}{\partial u}\delta u = (l + u(t))\cos\theta(t)\delta\theta + \sin\theta(t)\delta u \tag{6}$$

$$\delta y = \frac{\partial y}{\partial \theta}\delta\theta + \frac{\partial y}{\partial u}\delta u = (l + u(t))\sin\theta(t)\delta\theta - \cos\theta(t)\delta u \tag{7}$$

Principle of Virtual Displacements

Virtual displacements δr_i are any displacements consistent with the constraints of the system. The *principle of virtual displacements*[1] says that the work of real external forces through virtual external displacements equals the work of the real internal forces arising from the real external forces through virtual internal displacements consistent with the real external displacements. In system described by n coordinates r_i, with n internal inertial forces $m_i\ddot{r}_i(t)$, potential energy $V(\boldsymbol{r})$, and n external forces p_i *collocated* with coordinates r_i, the principle of virtual dispalcements says,

$$\sum_{i=1}^{n} \left(m_i\ddot{r}_i + \frac{\partial V}{\partial r_i} \right) \delta r_i = \sum_{i=1}^{n} (p_i(t))\, \delta r_i \tag{8}$$