Nalanda Open University

B.SC Part-1

Course : Physics(Hons)

Paper : 1

Prepared by : Dr Jaya Prakash Sinha – S.N.S College , Muzaffarpur. (BRABU).

Topic- Lagrange’s equations from d’Alembert’s principle (Classical mechanics)

Lagrange’s equations from d’Alembert’s principle

D’Alembert’s principle

The principle of virtual work states that the sum of the incremental virtual works done by all external forces $\boldsymbol{F}_i$ acting in conjunction with virtual displacements $\delta \boldsymbol{s}_i$ of the point on which the associated force is acting is zero:

$$\delta W = \sum_i \boldsymbol{F}_i \cdot \delta \boldsymbol{s}_i = 0. \tag{1.1.1}$$

This technique is useful for solving statics problems, with static forces of constraint. A static force of constraint is one that does no work on the system of interest, but merely holds a certain part of the system in place.

In a statics problem there are no accelerations. We can extend the principle of virtual work to dynamics problems, i.e., ones in which real motions and accelerations occur, by introducing the concept of inertial forces. For each parcel of matter in the system with mass m, Newton’s second law states that

$$\boldsymbol{F} = m\boldsymbol{a}. \tag{1.1.2}$$

We can make this dynamics problem look like a statics problem by defining an *inertial force*

$$\boldsymbol{F}^* = -m\boldsymbol{a} \tag{1.1.3}$$

and rewriting equation (1.1.2) as

$$\boldsymbol{F}_{total} = \boldsymbol{F} + \boldsymbol{F}^* = 0. \tag{1.1.4}$$

D'Alembert's principle is just the principle of virtual work with the inertial forces added to the list of forces that do work:

$$\delta W = \sum_i \boldsymbol{F}_i \cdot \delta \boldsymbol{s}_i + \sum_j \boldsymbol{F}_j^* \cdot \delta \boldsymbol{s}_j = 0. \tag{1.1.5}$$

Lagrange's equations from d'Alembert's principle

We begin with d'Alembert's principle written in its most fundamental and general form,

$$\sum_i (F_i + F_i^*)\delta x_i = 0 \tag{1.1}$$

where the subscript i ranges over all three components of all particles involved in the system of interest. The first step is to rewrite the particle positions, represented by the x_i in groups of three for each particle, in terms of independent *generalized coordinates* q_j. If there are constraints in the system, then there are fewer q variables than x variables. For example, a wheel rotating on a fixed axle has only one q, the angle of rotation, whereas there are three times as many x variables as there are atoms in the wheel.

For holonomic constraints we can write

$$x_i = x_i(q_j, t) \tag{1.2}$$

where we allow for the possibility that the relationship between the q and x variables to depend on time.

We can rewrite d'Alembert's principle by noting that

$$\delta x_i = \sum_j \frac{\partial x_i}{\partial q_j}\delta q_j \tag{1.3}$$

where the time dependence is not exercised since virtual changes are assumed to take place at a fixed time. Thus,

$$\sum_i F_i \delta x_i = \sum_{i,j} F_i \frac{\partial x_i}{\partial q_j} \delta q_j = \sum_j Q_j \delta q_j \tag{1.4}$$

where the

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j} \tag{1.5}$$

are called the *generalized forces*. Notice that just as the q_i need not have the units of length, the Q_i need not have the units of force. However, the product must have the units of energy. For instance, if q is a dimensionless angle, then the corresponding Q would be a torque, which has energy units.

Turning to the inertial forces in d'Alembert's principle, we note that

$$\sum_i F_i^* \delta x_i = -\sum_i m_i \frac{d^2 x_i}{dt^2} \delta x_i = -\sum_{i,j} m_i \frac{d^2 x_i}{dt^2} \frac{\partial x_i}{\partial q_j} \delta q_j \tag{1.6}$$

where we have used equation (1.3) in the last step. Using the product rule backwards, we see that

$$\sum_i m_i \frac{d^2 x_i}{dt^2} \frac{\partial x_i}{\partial q_j} = \sum_i m_i \left[\frac{d}{dt} \left(\dot{x}_i \frac{\partial x_i}{\partial q_j} \right) - \dot{x}_i \frac{d}{dt} \left(\frac{\partial x_i}{\partial q_j} \right) \right] \tag{1.7}$$

To make further progress, we take the total time derivative of equation (1.2), resulting in

$$\dot{x}_i = \sum_j \frac{\partial x_i}{\partial q_j}\dot{q}_j + \frac{\partial x_i}{\partial t} \tag{1.8}$$

where $\dot{x}_i = dx_i/dt$ and $\dot{q}_j = dq_j/dt$. The $\dot{x}_i$ are the actual particle velocity components. We call the $\dot{q}_j$ the *generalized velocities*. Taking the partial derivative of equation (1.8) with respect to a particular q_j, we immediately conclude that

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}, \tag{1.9}$$

where the derivative of the second term in equation (1.8) is zero because the velocities are not functions of the positions at an instant in time. (Ultimately, the positions can be derived from the velocities by integration, but this relationship depends on knowing the complete time history of the velocities, not just the values at a particular time.)

Finally, we note that

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right) = \frac{\partial \dot{x}_i}{\partial q_j}. \tag{1.10}$$

To show this, change the dummy summation index in equation (1.8) from j to k to avoid confusion, and take the partial derivative of this equation with respect to q_j:

$$\frac{\partial \dot{x}_i}{\partial q_j} = \sum_k \frac{\partial^2 x_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 x_i}{\partial q_j \partial t}. \tag{1.11}$$

This is possible again because $\dot{q}_k$ is not an explicit function of the q_j. Then compare this with

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right) = \sum_k \frac{\partial^2 x_i}{\partial q_k \partial q_j}\dot{q}_k + \frac{\partial^2 x_i}{\partial t \partial q_j}. \tag{1.12}$$

Aside from the order of partial derivatives, the right sides of equations (1.11) and (1.12) are identical, which proves equation (1.10).

Substituting equations (1.9) and (1.10) in equation (1.7) results in

$$\begin{aligned}
\sum_i m_i \frac{d^2 x_i}{dt^2}\frac{\partial x_i}{\partial q_j} &= \sum_i m_i \left[\frac{d}{dt}\left(\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_j}\right) - \dot{x}_i\left(\frac{\partial \dot{x}_i}{\partial q_j}\right)\right] \\
&= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\sum_i \frac{m_i \dot{x}_i^2}{2} - \frac{\partial}{\partial q_j}\sum_i \frac{m_i \dot{x}_i^2}{2} \\
&= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}
\end{aligned} \tag{1.13}$$

where we recognize the sums as the total kinetic energy T of the system.

Combining equations (1.4), (1.6), and (1.13) yields

$$\sum_j \left[Q_j - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) + \frac{\partial T}{\partial q_j}\right]\delta q_j = 0. \tag{1.14}$$

Since the q_i are independent of each other, the coefficients of the δq_i are individually zero, resulting in Lagrange's equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j. \tag{1.15}$$

Often forces are conservative and possible to represent as the gradient of a potential energy $F_i = -\partial V/\partial x_i$. Starting with the definition of generalized forces in equation (1.5), we find that

$$Q_j = \sum_i F_i \frac{\partial x_i}{\partial q_j} = -\sum_i \frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}. \tag{1.16}$$

If in addition, V is not an explicit function of time or of the generalized velocities, equation (1.15) may be written

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \tag{1.17}$$

where $L = T - V$ is called the *Lagrangian*. The lack of dependence on time and the generalized velocities allows the V to be incorporated in the first as well as the second terms of this equation. If some of the forces are conservative and others are not, then the more general form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \tag{1.18}$$

may be used.

The quantities

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{1.19}$$

are called the *generalized momenta*. Note that when the Lagrangian is not a function of a particular generalized coordinate and the associated non-conservative force Q_j is zero, then the associated generalized momentum is conserved, since equation (1.18) reduces to

$$\frac{dp_j}{dt} = 0. \tag{1.20}$$

To summarize, these equations are valid for systems obeying the following conditions:

1. The constraints on the system are holonomic, so that the q_j are independent for both finite and infinitesimal displacements. The constraints may be time dependent.

2. The potential energy V is a function only of the q_j. If there are forces for which no such potential exists, then they can be included on the right side of equation (1.18) in the Q_j.