Nalanda Open University

B.SC Part-3

Course : Physics(Hons)

Paper : 5

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Topic- Derivation of Lagrange's equations from Hamilton's principle

Derivation of Lagrange's equations from Hamilton's principle

Define a *Lagrangian* of kinetic and potential energies

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) \equiv T(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - V(\boldsymbol{q}, t)$$

and define an *action potential functional*

$$\mathcal{S}[\boldsymbol{q}(t)] = \int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\ dt$$

with end points $\boldsymbol{q}_1 = \boldsymbol{q}(t_1)$ and $\boldsymbol{q}_2 = \boldsymbol{q}(t_2)$. Consider the true path of $\boldsymbol{q}(t)$ from t_1 to t_2 and a variation of the path, $\delta\boldsymbol{q}(t)$ such that $\delta\boldsymbol{q}(t_1) = 0$ and $\delta\boldsymbol{q}(t_2) = 0$.

Hamilton's principle:[4]
The solution $\boldsymbol{q}(t)$ is an extremum of the action potential $\mathcal{S}[\boldsymbol{q}(t)]$
$\iff \delta\mathcal{S}[\boldsymbol{q}(t)] = 0$
$\iff \int_{t_1}^{t_2} \delta L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)\ dt = 0$
Substituting the Lagrangian into Hamilton's principle,

$$\int_{t_1}^{t_2} \left\{ \sum_i \left[\frac{\partial T}{\partial \dot{q}_i} \delta \dot{q}_i + \frac{\partial T}{\partial q_i} \delta q_i - \frac{\partial V}{\partial q_i} \delta q_i \right] \right\}\ dt = 0$$

We wish to factor out the independent variations δq_i, however the first term contains the variation of the derivative, $\delta \dot{q}_i$. If the conditions for admissible variations in position δq fully specify the conditions for admissible variations in velocity $\delta \dot{q}$, the variation and the differentiation can be transposed,

$$\frac{d}{dt}\delta q_i = \delta \dot{q}_i,$$

and we can integrate the first term by parts,

$$\int_{t_1}^{t_2} \frac{\partial T}{\partial \dot{q}_i}\delta \dot{q}_i = \left[\frac{\partial T}{\partial \dot{q}_i}\delta q_i\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\delta q_i \; dt$$

Since $\delta \boldsymbol{q}(t_1) = 0$ and $\delta \boldsymbol{q}(t_2) = 0$,

$$\int_{t_1}^{t_2} \left\{\sum_i \left[-\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\delta q_i + \frac{\partial T}{\partial q_i}\delta q_i - \frac{\partial V}{\partial q_i}\delta q_i\right]\right\} \; dt = 0$$

The variations δq_i must be arbitrary, so the term within the square brackets must be zero for all i.

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0.}$$