Nalanda Open University

B.SC Part-3

Course : Physics(Hons)

Paper : 5

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Topic- Derivation of Hamilton’s principle from d’Alembert’s principle

Derivation of Hamilton's principle from d'Alembert's principle

The variation of the potentential energy $V(\boldsymbol{r})$ may be expressed in terms of variations of the coordinates r_i

$$\delta V = \sum_{i=1}^{n} \frac{\partial V}{\partial r_i}\ \delta r_i = \sum_{i=1}^{n} f_i\ \delta r_i\ .$$

where f_i are potential forces collocated with coordiantes r_i. In Cartesian coordinates, the variation of the kinetic energy $T(\dot{\boldsymbol{r}})$

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{r}_i^2$$

may be expressed in terms of variations of the coordinate velocities, $\dot{r}_i$

$$\delta T = \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{r}_i}\ \delta \dot{r}_i = \sum_{i=1}^{n} m \dot{r}_i\ \delta \dot{r}_i\ .$$

For a system of n particle masses m_i acted on by n *internal* forces f_i of the potential V, d'Alembert's principle (?? or ??), is

$$\sum_{i=1}^{n} m_i \ddot{r}_i\ \delta r_i + \sum_{i=1}^{n} f_i\ \delta r_i = 0$$

Integrating d'Alembert's equation over a finite time period,

$$
\begin{aligned}
\int_{t_1}^{t_2} \sum_{i=1}^{n} m_i \ddot{r}_i \; \delta r_i \; dt + \int_{t_1}^{t_2} \sum_{i=1}^{n} f_i \; \delta r_i \; dt &= 0 \\
\sum_{i=1}^{n} \int_{t_1}^{t_2} m_i \frac{d}{dt} \dot{r}_i \; \delta r_i \; dt + \int_{t_1}^{t_2} \delta V \; dt &= 0 \\
\sum_{i=1}^{n} \left[m_i \dot{r}_i \; \delta r_i \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m_i \dot{r}_i \; \frac{d}{dt} \delta r_i \; dt \right] + \int_{t_1}^{t_2} \delta V \; dt &= 0 \\
- \int_{t_1}^{t_2} \sum_{i=1}^{n} m_i \dot{r}_i \; \delta \dot{r}_i \; dt + \int_{t_1}^{t_2} \delta V \; dt &= 0 \\
- \int_{t_1}^{t_2} \delta T \; dt + \int_{t_1}^{t_2} \delta V \; dt &= 0 \\
\delta \int_{t_1}^{t_2} (T - V) \; dt &= 0 \qquad (28)
\end{aligned}
$$

In this derivation we consider a variation of the coordinate motions $\delta r(t)$ from t_1 to t_2. That is, $\delta r(t_1) = 0$ and $\delta r(t_2) = 0$, which eliminates the first term in the third line. The fourth line involves a transposition of the variation and the derivitive $(d(\delta r)/dt = \delta \dot{r})$. The last line is a statement of Hamilton's principle, which is presented formally in the next section. Note that kinetic energy and potential energy are scalar valued quantities, invariant to changes in coordiante systems. So, while Hamilton's principle is derived here in the context of Cartesian coordinates, it applies to generalized coordinates, as well.