Nalanda Open University

B.SC Part-1

Course : Physics(Hons)

Paper : 1

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Topic- Application of Lagrange’s equations (Simple and Double Pendulum)

Application of Lagrange’s equations

Simple pendulum

We start with the simple pendulum, a problem that is easily solvable by elementary methods. Imagine the pendulum arm to have negligible mass and length D. A mass M is attached to the one end of the arm and the other is attached to a support that allows the pendulum to pivot freely in the x-z plane. The angle between the pendulum arm and the vertical is ϕ.

Taking zero elevation as the pivot point, the potential energy of the pendulum mass is

$$V = Mgz = -MgD\cos\phi. \tag{1.21}$$

Motion in the x-z plane is constrained to be in the form of a circular arc of radius D, which means that the kinetic energy of the pendulum is

$$T = \frac{1}{2}MD^2\dot{\phi}^2, \tag{1.22}$$

which means that the Lagrangian function is

$$L(\phi,\dot{\phi}) = T - V = \frac{1}{2}MD^2\dot{\phi}^2 + MgD\cos\phi. \tag{1.23}$$

The generalized coordinate (only one, as there is only one effective degree of freedom) is ϕ. Recall that ϕ and $\dot{\phi}$ are considered to be independent variables in Lagrangian dynamics, so

$$\frac{\partial L}{\partial \dot{\phi}} = MD^2\dot{\phi} \tag{1.24}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = MD^2\frac{d^2\phi}{dt^2}. \tag{1.25}$$

The second term in Lagrange's equation is also easily calculated:

$$\frac{\partial L}{\partial \phi} = -MgD\sin\phi. \tag{1.26}$$

Combining equations (1.25) and (1.26), we arrive at the governing equation for the pendulum:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = MD^2\frac{d^2\phi}{dt^2} + MgD\sin\phi = 0 \tag{1.27}$$

which simplifies to the usual form

$$\frac{d^2\phi}{dt^2} + \frac{g}{D}\sin\phi = 0. \tag{1.28}$$

For small ϕ we have $\sin\phi \approx \phi$ and we have a harmonic oscillator equation for the pendulum.

Double pendulum

A particular form of the double pendulum is illustrated in figure 1.1. The masses are free to swing in the x-z plane, with the second pendulum swinging from the bob on the first pendulum. (Other forms include a version with massive rods instead of weights attached to rods of negligible mass.)

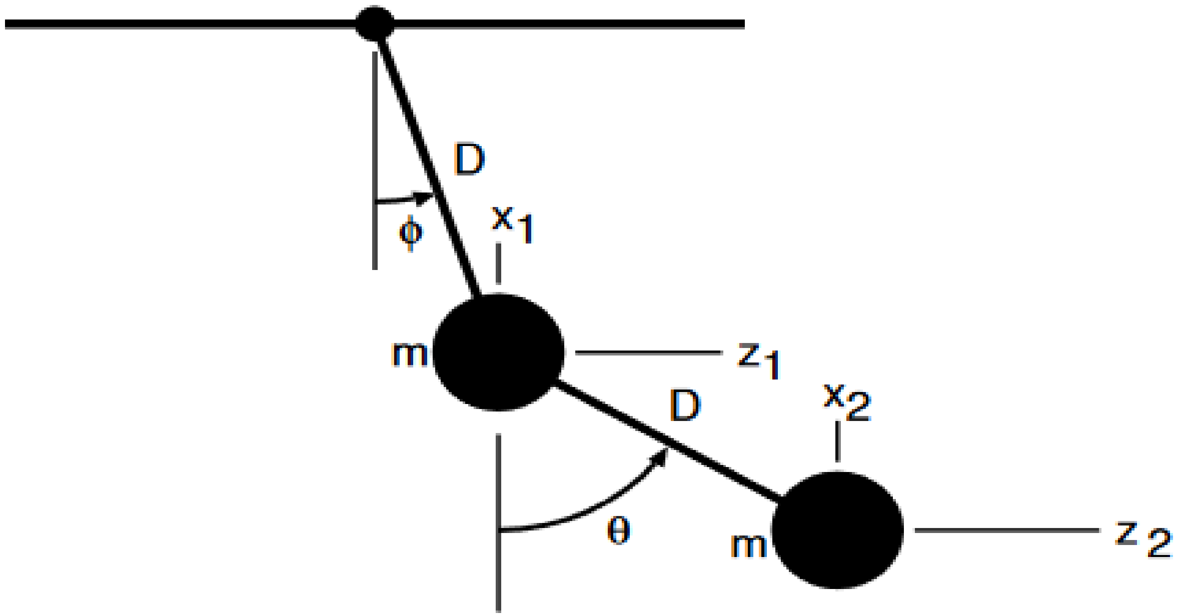

Figure 1.1: Sketch for the double pendulum.

The Cartesian coordinates of the two masses are related to the angles ϕ and θ as follows

$$(x_1, z_1) = (D\sin\phi, -D\sin\phi) \tag{1.29}$$

and

$$(x_2, z_2) = [D(\sin\phi + \sin\theta), -D(\cos\phi + \cos\theta) \tag{1.30}$$

where the origin of the coordinate system is located where the pendulum attaches to the ceiling. The kinetic energies of the two pendulums are

$$\begin{aligned} T_1 =& \frac{1}{2}m(\dot{x}_1^2 + \dot{z}_1^2) \\ =& \frac{1}{2}mD^2\dot{\phi}^2 \end{aligned} \tag{1.31}$$

$$\begin{aligned} T_2 =& \frac{1}{2}m(\dot{x}_2^2 + \dot{z}_2^2) \\ =& \frac{1}{2}mD^2\left[\dot{\phi}^2 + \dot{\theta}^2 + 2\cos(\phi - \theta)\dot{\phi}\dot{\theta}\right] \end{aligned} \tag{1.32}$$

and the potential energy of the two pendulum bobs together is

$$V = mg(z_1 + z_2) = -mgD\,(2\cos\phi + \cos\theta)\,. \tag{1.33}$$

The angles ϕ and θ and their time derivatives are the generalized coordinates and velocities and the Lagrangian is

$$L = \frac{1}{2}mD^2\left[2\dot{\phi}^2 + \dot{\theta}^2 + 2\cos(\phi-\theta)\dot{\phi}\dot{\theta}\right] + mgD\,(2\cos\phi + \cos\theta)\,. \tag{1.34}$$

From

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{1.35}$$

we get the governing equations

$$2\frac{d^2\phi}{dt^2} + \cos(\phi-\theta)\frac{d^2\theta}{dt^2} + \sin(\phi-\theta)\dot{\theta}^2 + \frac{2g}{D}\sin\phi = 0 \tag{1.36}$$

and

$$\frac{d^2\theta}{dt^2} + \cos(\phi-\theta)\frac{d^2\phi}{dt^2} - \sin(\phi-\theta)\dot{\phi}^2 + \frac{g}{D}\sin\theta = 0. \tag{1.37}$$

Thus, two degrees of freedom result in two coupled governing equations. These happen to be complicated nonlinear equations and the physical system demonstrates chaotic behavior for large amplitude oscillations.