Nalanda Open University

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Topic- Quantum Mechanics (Heisenberg's Uncertainty Principle)

Heisenberg’s Uncertainty Principle

According to the analysis contained in the previous two sections, a particle wave packet which is initially localized in x-space with characteristic width Δx is also localized in k-space with characteristic width $\Delta k = 1/(2\,\Delta x)$. However, as time progresses, the width of the wave packet in x-space increases, whilst that of the wave packet in k-space stays the same.

$$\Delta x\,\Delta k \gtrsim \frac{1}{2}. \tag{1}$$

Furthermore, we can think of Δx and Δk as characterizing our *uncertainty* regarding the values of the particle’s position and wavenumber, respectively.

Now, a measurement of a particle’s wavenumber, k, is equivalent to a measurement of its momentum, p, since $p = \hbar\,k$. Hence, an uncertainty in k of order Δk translates to an uncertainty in p of order $\Delta p = \hbar\,\Delta k$. It follows from the above inequality that

$$\Delta x\,\Delta p \gtrsim \frac{\hbar}{2}. \tag{2}$$

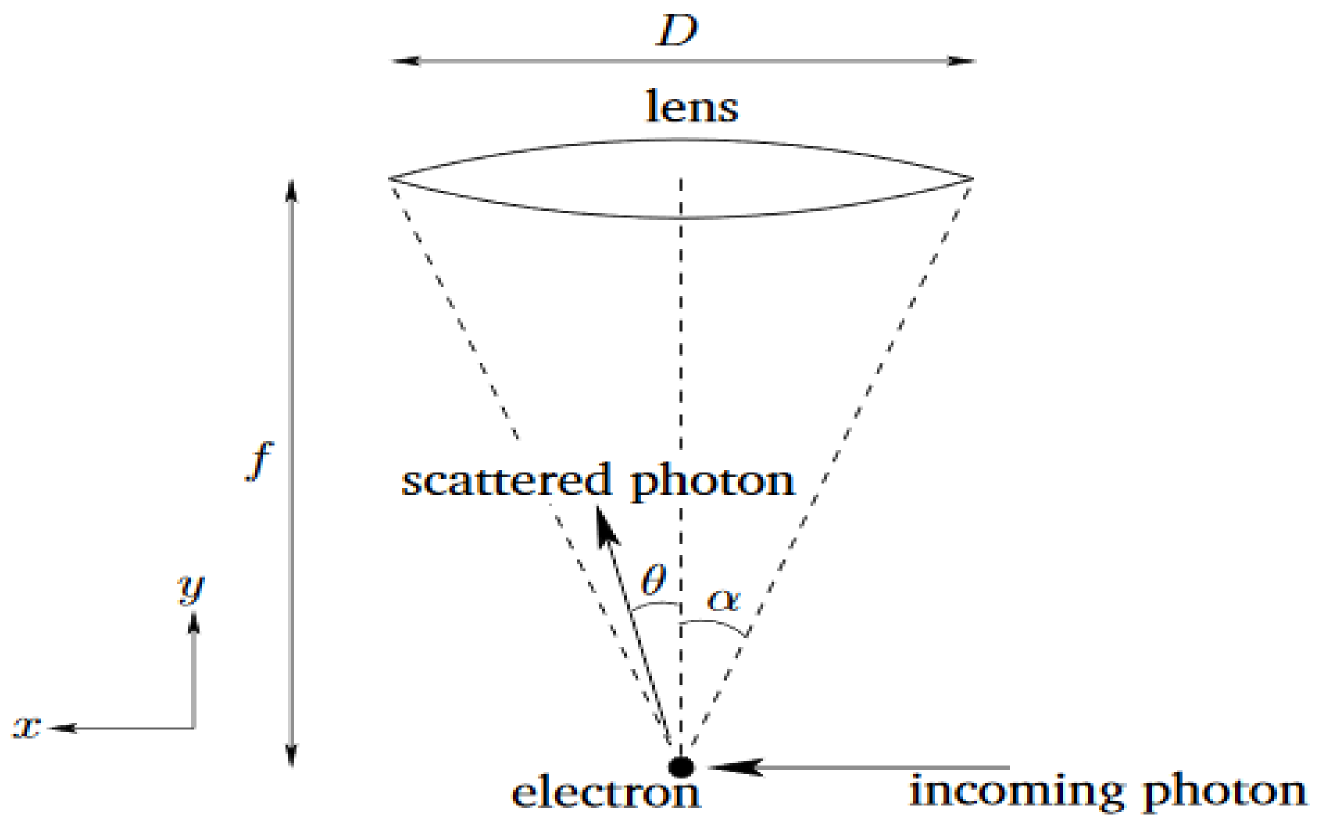

Figure (1) *Heisenberg's microscope.*

This is the famous *Heisenberg uncertainty principle*, first proposed by Werner Heisenberg in 1927. According to this principle, it is impossible to simultaneously measure the position and momentum of a particle (exactly). Indeed, a good knowledge of the particle's position implies a poor knowledge of its momentum, and *vice versa*. Note that the uncertainty principle is a direct consequence of representing particles as waves.

It can be seen from Eqs. **& theretical representations** that at large t a particle wavefunction of original width Δx (at $t = 0$) spreads out such that its spatial extent becomes

$$\sigma \sim \frac{\hbar\, t}{m\, \Delta x}. \tag{3}$$

It is easily demonstrated that this spreading is a consequence of the uncertainty principle. Since the initial uncertainty in the particle's position is Δx, it follows that the uncertainty in its momentum is of order $\hbar/\Delta x$. This translates to an uncertainty in velocity of $\Delta v = \hbar/(m\,\Delta x)$. Thus, if we imagine that parts of the wavefunction propagate at $v_0 + \Delta v/2$, and others at $v_0 - \Delta v/2$, where v_0 is the mean propagation velocity, then the wavefunction will spread as time progresses. Indeed, at large t we expect the width of the wavefunction to be

$$\sigma \sim \Delta v\, t \sim \frac{\hbar\, t}{m\, \Delta x}, \tag{4}$$

which is identical to Eq. (3) . Evidently, the spreading of a particle wavefunction must be interpreted as an increase in our *uncertainty* regarding the particle's position, rather than an increase in the spatial extent of the particle itself.

Figure (1) illustrates a famous thought experiment known as *Heisenberg's microscope*. Suppose that we try to image an electron using a simple optical system in which the objective lens is of diameter D and focal-length f. (In practice, this would only be possible using extremely short wavelength light.) It is a well-known result in optics that such a system has a minimum angular resolving power of λ/D, where λ is the wavelength of the light illuminating the electron. If the electron is placed at the focus of the lens, which is where the minimum resolving power is achieved, then this translates to a uncertainty in the electron's transverse position of

$$\Delta x \simeq f\,\frac{\lambda}{D}. \tag{5}$$

However,

$$\tan\alpha = \frac{D/2}{f}, \tag{6}$$

where α is the half-angle subtended by the lens at the electron. Assuming that α is small, we can write

$$\alpha \simeq \frac{D}{2\,f}, \tag{7}$$

so

$$\Delta x \simeq \frac{\lambda}{2\,\alpha}. \tag{8}$$

It follows that we can reduce the uncertainty in the electron's position by *minimizing* the ratio λ/α: *i.e.*, by using short wavelength radiation, and a wide-angle lens.

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Let us now examine Heisenberg's microscope from a quantum mechanical point of view. According to quantum mechanics, the electron is imaged when it scatters an incoming photon towards the objective lens. Let the wavevector of the incoming photon have the (x, y) components $(k, 0)$. See Fig. (1) If the scattered photon subtends an angle θ with the center-line of the optical system, as shown in the figure, then its wavevector is written $(k\sin\theta, k\cos\theta)$. Here, we are ignoring any wavelength shift of the photon on scattering—*i.e.*, the magnitude of the $\mathbf{k}$-vector is assumed to be the same before and after scattering. Thus, the change in the x-component of the photon's wavevector is $\Delta k_x = k(\sin\theta - 1)$. This translates to a change in the photon's x-component of momentum of $\Delta p_x = \hbar k(\sin\theta - 1)$. By momentum conservation, the electron's x-momentum will change by an equal and opposite amount. However, θ can range all the way from $-\alpha$ to $+\alpha$, and the scattered photon will still be collected by the imaging system. It follows that the uncertainty in the electron's momentum is

$$\Delta p \simeq 2\hbar k\sin\alpha \simeq \frac{4\pi\hbar\alpha}{\lambda}. \tag{9}$$

Note that in order to reduce the uncertainty in the momentum we need to *maximize* the ratio λ/α. This is exactly the opposite of what we need to do to reduce the uncertainty in the position. Multiplying the previous two equations, we obtain

$$\Delta x\, \Delta p \sim h, \tag{10}$$

which is essentially the uncertainty principle.

According to Heisenberg's microscope, the uncertainty principle follows from two facts. First, it is impossible to measure any property of a microscopic dynamical system without *disturbing* the system somewhat. Second, particle and light energy and momentum are *quantized*. Hence, there is a limit to how small we can make the aforementioned disturbance. Thus, there is an irreducible uncertainty in certain measurements which is a consequence of the act of measurement itself.